

TRANSCENDENTAL TRACE FORMULAS FOR FINITE-GAP POTENTIALS

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ABSTRACT. We show that formulas differing from classical analogues of rational trace formulas for algebraic-geometric potentials occur in the theory of finite-gap integration of spectral equations. The new formulas contain transcendental modular functions and hypergeometric series. They result in transcendental relations for theta functions.

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1. INTRODUCTION

Trace identities and formulas are known in the integration theory of spectral problems. This terminology originates from papers by Krein, Gel'fand, and Dickey on the spectral theory of the ordinary differential equation of the form $\psi_{xx} - u(x)\psi = \lambda\psi$ dating back to the 1950s. In 1975, stimulated by the appearance of finite-gap integration methods (in the spectral formulation), V. Matveev established a new and important interpretation of these formulas. These relations, together with their t -isospectral modifications, are differential identities for exactly solvable potentials $u(x)$. In particular, using one of the series of these formulas, namely, the famous formula

$$u(x) = 2\{\gamma_1(x) + \cdots + \gamma_g(x)\} + \text{const}, \quad (1)$$

allows reconstructing the potentials from the quantities $\gamma_k(x)$, which satisfy integrable ordinary differential equations that are the Dubrovin equations. All the remaining trace identities follow from this “master” formula. Here, we interpret these relations or, to be precise, their analogues for higher-order spectral equations as formulas that express an exactly solvable potential in terms of the quantities $\gamma_k(x)$.

Key words and phrases. Spectral problem, finite-gap potential, trace formula, modular functions, theta-functions, elliptic functions.

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It is known that a potential $u(x)$ determines a Liouville-integrable finite-dimensional Hamiltonian system in which x plays the role of time. We pass to the canonical variables (p, q) by combining the potential and its derivatives: $p = p(u, u_x, \dots)$, $q = q(u, u_x, \dots)$ [8]. The subsequent transition to separation variables is the transition to the variables γ_k and μ_k subject to the algebraic relation $W(\gamma_k, \mu_k) = 0$, which is called the spectral curve equation. Formulas of the inverse transition $p = p(\gamma_k, \mu_k)$, $q = q(\gamma_k, \mu_k)$ and hence the formula of the form $u = u(p, q)$ are well known [2] in Hamiltonian system theory, but we can clearly identify formula (1) with a differential analogue of an operator trace only in the Schrödinger equation case. A “trace” interpretation of all other algebraic-geometric (finite-gap) operators is lacking. This is because both the spectral curve coordinates γ and μ must participate in a general theory of variable separation because Abelian functions, generally speaking, are expressed in terms of both these variables. Second-order equations are the only particular case where $u = u(\gamma)$. Only isolated examples are known for more complex spectral problems [4], while a general scheme for constructing the principal trace formula

$$u = R(\gamma_k, \mu_k) \quad (2)$$

with a rational function R is still lacking, to the best of our knowledge.

We devote this paper not to this problem but to the observation that in addition to representing potentials in the form of Abelian functions as in (2), we have other representations. They are provided by transcendental, not rational but still *single-valued*, functions of symmetric combinations of the parameters (γ_k, μ_k) . The example we use to demonstrate this is interesting not only in itself but also because it leads to new transcendental identities for Θ -functions. We recall that polynomial identities for Θ -functions are well known and are in fact uniformizing representations for Abelian varieties [3] and Θ -functional representations for Abelian functions (2) respectively.

2. THE RATIONAL TRACE FORMULA

We consider the third-order spectral problem

$$\Psi''' + u(x)\Psi' - \frac{1}{2}u'(x)\Psi = \lambda\Psi. \quad (3)$$

It is known that if we associate this problem with a nonlinear integrable equation, then its isospectral deformations are described by the Kaup–Kupershmidt equation [11]

$$u_t = u_{xxxxx} + 5uu_{xxx} + \frac{25}{2}u_xu_{xx} + 5u^2u_x. \quad (4)$$

In what follows, it suffices to consider only stationary solutions or, equivalently, the operator pencil

$$A(\lambda; [u])\Psi'' + B(\lambda; [u])\Psi' + C(\lambda; [u])\Psi = \mu\Psi, \quad (5)$$

which depends linearly on λ and commutes with (3).

2.1. The algebraic curve. The compatibility condition of Eq. (5) with the spectral problem, i. e., with problem (3), customarily yields expressions for A , B , and C :

$$A = -9\lambda, \quad B = u^2 + \frac{1}{2}u'' + c, \quad C = -\frac{1}{3}A'' + \frac{2}{3}Au - B',$$

The constant is related to the stationary dynamics $u = u(x - ct)$, and we have

$$-9\lambda\Psi'' + \left(u^2 + \frac{1}{2}u'' + c\right)\Psi' - \left(6u\lambda + 2uu'' + \frac{1}{2}u'''\right)\Psi = \mu\Psi.$$

The quantities λ and μ are algebraically dependent, i.e., they belong to the algebraic curve

$$\mu^3 + Q(\lambda; [u])\mu + T(\lambda; [u]) = 0. \quad (6)$$

We obtain the equation of this curve by eliminating Ψ from (3) and (5), with the result

$$\mu^3 + \left(27c\lambda^2 - \frac{9}{2}K\lambda + E_2\right)\mu + 729\lambda^5 + E_1\lambda^3 + \frac{27}{2}uK\lambda^2 + E_3\lambda + \frac{1}{8}(u'' + 2u^2 + 2c)^2K = 0.$$

We present expressions for E_k below and here note that not only integrals of stationary equation (4) but also the equation itself enter the spectral curve equation:

$$K = cu' + u^{(\vee)} + 5uu''' + \frac{25}{2}u'u'' + 5u^2u'.$$

By virtue of (4) the constant K must vanish identically. But the expression $u'' + 2u^2 + 2c = \text{const}$ also does not contradict it. If the constant in the right-hand side of this equation is nonzero, then the compatibility with the equation $K = 0$ results in the trivial answer $u = \text{const}$. Otherwise, we have the degenerate curve

$$\mu^3 + 27c\lambda^2\mu + 729\lambda^5 - 27\left(\frac{3}{4}u'^2 + u^3 + 3cu\right)\lambda^3 = 0.$$

This curve has the genus $g = 1$, the corresponding solution $u = -3\wp(x + \frac{3}{4}g_2t)$ can be easily calculated, and we are not interested in it. In the nondegenerate case, curve (6) becomes

$$\mu^3 + (27c\lambda^2 + E_2)\mu + 729\lambda^5 + 81E_1\lambda^3 + E_3\lambda = 0, \quad (7)$$

it has the genus $g = 4$, and the problem of determining the corresponding potential is nontrivial.

2.2. The rational trace formula. We first obtain the standard trace formula, i.e., the representation of the potential in form (2). Let us determine the quantities γ_k . In the finite-gap potential theory, these objects are the poles of $\Psi(x)$, which are independent of λ . Eliminating the Ψ -function and its first derivative from Eqs. (3) and (5), we obtain the expression

$$\frac{\Psi'}{\Psi} = 27\lambda \frac{3\mu^2 + Q}{\Pi} - \frac{1}{18\lambda}(u'' + 2u^2 + 2c). \quad (8)$$

Here $Q(\lambda) := 27c\lambda^2 + E_1([u])$, and the differential polynomial $\Pi = \Pi(\lambda; [u])$ is

$$\begin{aligned} 3^{-7}\Pi(\lambda; [u]) &= \lambda^4 - \frac{1}{6}u'\lambda^3 + \frac{1}{18}(u^{(\vee)} + 5uu'' + 4u'^2 + 2u^3 + 2cu)\lambda^2 + \\ &+ \frac{1}{324}(u''' + 4uu')(u'' + 2u^2 + 2c)\lambda + \frac{1}{3^6 8}(u'' + 2u^2 + 2c)^3. \end{aligned} \quad (9)$$

We now present the Novikov integrals E_k :

$$E_1 = u^{(\vee)} + 5uu'' + \frac{15}{4}u'^2 + \frac{5}{3}u^3 + cu, \quad (10)$$

$$E_2 = -\frac{1}{4}u'''(u''' + 8uu') - \frac{1}{4}uu''^2 + \frac{1}{24}u''(E_1 + 3u'^2 - 20u^3 - 12cu) + \dots,$$

$$E_3 = -\frac{3}{4}u'''u'(u'' + 2u^2 + 2c) + \frac{1}{4}u''^3 + \frac{3}{4}(u^2 + c)u''^2 - 3u'^2u'' + \dots,$$

where we truncate the expressions for E_2 , E_3 because we do not need them in what follows.

The separation variables γ_k determine the poles of logarithmic derivative (8), i.e., they factor the polynomial Π . We therefore define

$$\Pi = 3^7(\lambda - \gamma_1)(\lambda - \gamma_2)(\lambda - \gamma_3)(\lambda - \gamma_4). \quad (11)$$

If we define the second coordinate $\mu_k = \mu_k(x)$ by the formula

$$\mu_k^3 + Q(\gamma_k(x))\mu_k + T(\gamma_k(x)) = 0,$$

then we obtain an analogue of the Dubrovin equations from (8) in the case under consideration. Indeed, passing to the limit $\lambda \rightarrow \gamma_k$ in (8), we obtain

$$\frac{d\gamma_k}{dx} = -\frac{\gamma_k}{81} \frac{3\mu_k^2 + Q(\gamma_k)}{\prod_{j \neq k} (\gamma_k - \gamma_j)}. \quad (12)$$

Additional explanations can be found in [4], where the general method can also be found for obtaining such equations and the second coordinate of the zero of the polynomial Π :

$$\mu_k = 3u\gamma_k + \frac{(u'' + 2u^2 + 2c)^2}{36\gamma_k}. \quad (13)$$

Comparing (9) and (11), for instance, we obtain the identities

$$u' = 6 \sum_1^4 \gamma_k, \quad u^{(\text{IV})} + 5uu'' + 4u'^2 + 2u^3 + 2cu = 18 \sum_{k>j} \gamma_k \gamma_j. \quad (14)$$

Composing symmetric combinations of Eqs. (12) and using formula (13) for the second coordinate, we can eliminate all the derivatives of the potential and thus obtain a linear equation for u . This is the desired trace formula. Omitting the details, we eventually obtain

$$u = \frac{1}{3} \sum_1^4 \frac{\gamma_k^2 \mu_k}{\prod_{j \neq k} (\gamma_k - \gamma_j)}. \quad (15)$$

The last simplification of Eqs. (12) is to reduce them to quadratures. Again composing symmetric combinations of the right-hand sides of (12), we obtain the equations

$$\begin{aligned} \sum_1^4 \frac{d\gamma_k}{3\mu_k^2 + Q(\gamma_k)} &= 0, & \sum_1^4 \frac{\gamma_k d\gamma_k}{3\mu_k^2 + Q(\gamma_k)} &= 0, \\ \sum_1^4 \frac{\mu_k d\gamma_k}{3\mu_k^2 + Q(\gamma_k)} &= 0, & \sum_1^4 \frac{\gamma_k^2 d\gamma_k}{3\mu_k^2 + Q(\gamma_k)} &= -\frac{1}{81} dx, \end{aligned}$$

which, as expected, result in a Jacobi problem, i.e., in the problem of inverting four holomorphic integrals on curve (7):

$$\begin{aligned} \sum_1^4 \int_{(\gamma_k, \mu_k)} \frac{d\lambda}{3\mu^2 + Q(\lambda)} &= a_1, & \sum_1^4 \int_{(\gamma_k, \mu_k)} \frac{\lambda d\lambda}{3\mu^2 + Q(\lambda)} &= a_2, \\ \sum_1^4 \int_{(\gamma_k, \mu_k)} \frac{\mu d\lambda}{3\mu^2 + Q(\lambda)} &= a_3, & \sum_1^4 \int_{(\gamma_k, \mu_k)} \frac{\lambda^2 d\lambda}{3\mu^2 + Q(\lambda)} &= a_4 - \frac{1}{81} x. \end{aligned}$$

In what follows, we consider the dependence $\gamma_k = \gamma_k(x)$ to be defined from these equations.

We note that this scheme for obtaining the Dubrovin equation and trace formula from the definition of the differential polynomial Π and Ψ is intrinsic for general finite-gap operators. The definitions of the quantities γ_k , the Dubrovin equations, and various relations for the potential and its derivatives (different versions of the trace formula) are “simultaneously entangled” in the general set of determining polynomial relations.

3. THE TRANSCENDENTAL TRACE FORMULA

Deriving the preceding formulas, we manipulated with polynomials, which is common in polynomial ideal theory [7]. The bases of such ideals are not unique, which means that we can presumably find other relations determining the potential. This is indeed the case. Eliminating the higher derivative $u^{(\text{iv})}$ from the Novikov integral E_1 and from the second identity in (10), we obtain the quadratic equation for u' :

$$3u'^2 + 4u^3 + 12cu + E_1 = 216 \sum_{k>j} \gamma_k \gamma_j.$$

But u' can be expressed uniquely in terms of γ by virtue of the first identity in (14). We then obtain the cubic polynomial in the variable u :

$$u^3 + 3cu + 3E_1 + 27 \sum_{k=1}^4 \gamma_k^2 = 0. \quad (16)$$

It can be interpreted as an algebraic (implicit) variant of the principal trace formula.

On the other hand, finite-gap potentials are single-valued functions of the variable x . They satisfy autonomous ordinary differential equations, pass the Painlevé test, and are expressed in terms of theta functions. It is also known (but seldom used) that roots of any 3rd or 4th degree polynomial can be expressed analytically via its coefficients in terms of elliptic functions [1]. For exhaustive information on this topic see [5]. In our case the situation simplifies since we may directly consider polynomial (16) as a Weierstrass cubic polynomial

$$\begin{aligned} 4u^3 + 12cu + 12E_1 + 108 \sum_{k=1}^4 \gamma_k^2(x) &= 4u^3 - au - b = \\ &= 4(u - e)(u - e')(u - e''), \end{aligned}$$

where the points e , e' , and e'' depend on the coefficients a and b , which in turn depend on x . The roots of this polynomial are

$$u = \{e, e', e''\} = \wp(\omega_k(x); a, b),$$

where

$$a = -12c, \quad b = -108 \sum \gamma_k^2(x) - 12E_1,$$

are expressed in terms of the theta constants [1]

$$\begin{aligned} e &= \frac{1}{\omega^2} \frac{\pi^2}{12} \{\vartheta_3^4(\tau) + \vartheta_4^4(\tau)\}, \\ e' &= -\frac{1}{\omega^2} \frac{\pi^2}{12} \{\vartheta_2^4(\tau) + \vartheta_3^4(\tau)\}, \\ e'' &= \frac{1}{\omega^2} \frac{\pi^2}{12} \{\vartheta_2^4(\tau) - \vartheta_4^4(\tau)\}, \end{aligned} \quad (17)$$

and the quantities ω and $\tau = \omega'/\omega$ are expressed in terms of the coefficients a and b . It suffices to take just one such expression and express the ϑ -constants in (17) using the classic series [1]:

$$\vartheta_2(\tau) := e^{\frac{\pi}{4}i\tau} \sum_{k=-\infty}^{\infty} e^{(k^2+k)\pi i\tau}, \quad \vartheta_3(\tau) := \sum_{k=-\infty}^{\infty} e^{k^2\pi i\tau}, \quad \vartheta_4(\tau) := \sum_{k=-\infty}^{\infty} (-1)^k e^{k^2\pi i\tau}.$$

(we let the symbol e^z denote the exponential function to distinguish it from the Weierstrass e -points).

We thus obtain a classic elliptic modular function inversion problem, i. e., the problem of expressing the periods $(2\omega, 2\omega')$ of the elliptic curve $w^2 = 4z^3 - az - b$ in terms of its coefficients. The solution scheme is known. We must find the root τ of the transcendental equation

$$J(\tau) = \frac{a^3}{a^3 - 27b^2},$$

where $J(\tau)$ is the classic modular Klein function [1, 10]. After finding the root τ , we calculate the half-periods (ω, ω') by the formula

$$\omega^2 = \frac{a}{b} \frac{g_3(\tau)}{g_2(\tau)}, \quad \omega' = \tau\omega, \quad (18)$$

where $g_{2,3}(\tau)$ are the known modular forms [10]. We have the Eisenstein series, the Hurwitz–Lambert series, or the ϑ -constant representations for these forms:

$$\begin{aligned} g_2(\tau) &= \frac{\pi^4}{24} \{ \vartheta_2^8(\tau) + \vartheta_3^8(\tau) + \vartheta_4^8(\tau) \} \\ g_3(\tau) &= \frac{\pi^6}{432} \{ \vartheta_2^4(\tau) + \vartheta_3^4(\tau) \} \{ \vartheta_3^4(\tau) + \vartheta_4^4(\tau) \} \{ \vartheta_4^4(\tau) - \vartheta_2^4(\tau) \}. \end{aligned} \quad (19)$$

It remains to write the expression for the root τ . Strangely enough, an explicit analytic solution of this classic problem is lacking¹, although it is common knowledge that this solution can be written in terms of ratios of hypergeometric ${}_2F_1$ -series. Because the series ${}_2F_1(J)$ converges only inside the unit circle, the formulas differ depending on whether $|J| > 1$ or $|J| < 1$. Altogether, this results in cumbersome expressions also containing functional series in the logarithmic derivative of the Euler Γ -function. For example, the corresponding expression (22)–(27) in Sec. 14.6.2 of [10] is half a page long (even if we forgive the incorrectness of formula (23)). But the problem admits a simple solution.

We use the well-known fact that the function J is closely related to the hypergeometric equation of the form

$$J(J-1)\psi'' + \frac{1}{6}(7J-4)\psi' + \frac{1}{144}\psi = 0.$$

Solutions of general hypergeometric equations are hypergeometric functions ${}_2F_1(a, b; c|z)$, but these functions can be rewritten in terms of classic special functions under special restrictions imposed on the parameters a , b , and c . We mean not reductions of the function ${}_2F_1$ but the cases where the ${}_2F_1$ -series admits a quadratic transformation, and the hypergeometric equation then reduces to a two-parameter equation, e. g., to the Legendre equation. This is true for the above equation. Its solution is a linear combination

$$\psi = \sqrt[6]{J} \{ AP_\nu^\mu(\sqrt{1-J}) + BQ_\nu^\mu(\sqrt{1-J}) \}$$

¹Unfortunately, solutions of this problem are given incorrectly both on page 789 of encyclopedia [6] and in the end of Sect. 11 of the remarkable book [1]; this is not due to misprints.

of the Legendre functions with the parameters $(\nu, \mu) = (-\frac{1}{2}, \frac{1}{3})$ (see [9] for detailed information about these functions). From the above, we conclude that the formula

$$\tau = \frac{aP(\sqrt{1-J}) + bQ(\sqrt{1-J})}{cP(\sqrt{1-J}) + dQ(\sqrt{1-J})} \quad (20)$$

holds at certain numerical values of $\{a, b, c, d\}$ (we omit the indices ν, μ of the Legendre functions for brevity). We present only the final answer; see [5] for derivation.

Proposition 1. *For the elliptic curve $w^2 = 4z^3 - az - b$, the period ratio $\tau = \omega'/\omega$ is*

$$\tau = \left\{ \pi i \frac{P(\sqrt{1-J})}{Q(\sqrt{1-J})} - 1 \right\} e^{\frac{\pi}{3}i}, \quad \text{where } J := \frac{a^3}{a^3 - 27b^2}. \quad (21)$$

Remark 1. If J is substituted for $J(\tau)$ in equality (21), then this equality becomes an identity that holds for all τ in the upper half-plane \mathbb{H}^+ . The left- and right-hand sides of this identity must be calculated up to the action of the total modular group $\text{PSL}_2(\mathbb{Z}) =: \Gamma(1)$. Relation (21) is therefore a $\Gamma(1)$ -equivalent of the solution of the modular inversion problem in the Legendre representation. Namely, given the elliptic curve in the Legendre form $w^2 = (1 - z^2)(1 - k^2 z^2)$, we can calculate the elliptic modulus τ by the celebrated Jacobi formula

$$\tau = i \frac{K'(k)}{K(k)} \pmod{\Gamma(2)}, \quad (22)$$

where K and K' are the complete Legendre elliptic integrals of the first kind [1, 10, 13].

Returning to the potential, we use formulas (17) and (18) with (19) taken into account to obtain the desired transcendental “trace formula”:

$$u(x) = \frac{3}{2c} \left\{ E_1 + 9 \sum_k^4 \gamma_k^2(x) \right\} \frac{\vartheta_2^8(\tau) + \vartheta_3^8(\tau) + \vartheta_4^8(\tau)}{\{\vartheta_2^4(\tau) + \vartheta_3^4(\tau)\} \{\vartheta_4^4(\tau) - \vartheta_2^4(\tau)\}}, \quad (23)$$

where τ is the function

$$\tau(x) = \hat{\Gamma} \left(\pi i \frac{P(\sqrt{1-J})}{Q(\sqrt{1-J})} e^{\frac{\pi}{3}i} - e^{\frac{\pi}{3}i} \right), \quad J := \frac{4c^3}{4c^3 + 9 \{E_1 + 9 \sum \gamma_k^2(x)\}^2}, \quad (24)$$

and we let the symbol $\hat{\Gamma}$ denote the operation of setting a number to the fundamental domain of the group. Formulas (23)–(24) do not contain the coordinates μ_k . Verifying that function (23) indeed satisfies relation (10) or a stationary version of Eq. (4) is a good and rather nontrivial exercise.

4. TRANSCENDENTAL IDENTITIES FOR Θ -FUNCTIONS

The results in the preceding section demonstrate that we can analogously obtain many more examples of Abelian functions that are radicals or roots of algebraic equations. For example, the last summand in polynomial (9) provides a single-valued representation of the Abelian function radical of the form

$$18 \sqrt[3]{\gamma_1 \gamma_2 \gamma_3 \gamma_4} = u'' + 2u^2 + 2c,$$

and the general mechanism for constructing “transcendental traces” is based on eliminating derivatives of the potential from the total set of relations determining the integrals E_k and the variables γ_k . Solving the obtained equations in single-valued functions (which are

always theta functions), we obtain the answer. These equations (and even possibly their orders) are not uniquely defined and may depend on the genus, but transitions between them are just different representations of the same Abelian function. They differ even in the (γ, μ) -representation because the coordinates γ and μ are algebraically related.

Now equating the transcendental and rational representations, we obtain an unusual identity containing theta functions. Indeed, comparing Eqs. (15) and (23), for example, (23)

$$\frac{1}{3} \sum_k \frac{\gamma_k^2 \mu_k}{\prod_{j \neq k} (\gamma_k - \gamma_j)} = \frac{3}{2c} \left\{ E_1 + 9 \sum_k \gamma_k^2 \right\} \frac{\vartheta_2^8(\tau) + \vartheta_3^8(\tau) + \vartheta_4^8(\tau)}{\{\vartheta_2^4(\tau) + \vartheta_3^4(\tau)\} \{\vartheta_4^4(\tau) - \vartheta_2^4(\tau)\}}, \quad (25)$$

we can in principle represent the Abelian functions rational in (γ, μ) in this formula in the form of Θ -function relations [3]. The obtained relation for Θ -functions of genus $g = 4$ is a transcendental identity containing the theta constants $\vartheta(\tau(x))$, although curve (7), generally speaking, is not related to any elliptic curve. In addition to the curve moduli, this identity also contains an arbitrary parameter x . We can also move all symmetric functions that are rational in (γ, μ) to the left-hand side of the equality. We then obtain an Abelian function for the Jacobian of trigonal curve (7) in the left-hand side and an “exotic” representation of this curve in terms of the elliptic theta constants of the Legendre functions in the right-hand side:

$$u^2 = -2c \frac{\{\vartheta_2^4(\tau) + \vartheta_3^4(\tau)\} \{\vartheta_4^4(\tau) - \vartheta_2^4(\tau)\}}{\vartheta_2^8(\tau) + \vartheta_3^8(\tau) + \vartheta_4^8(\tau)} - 3c. \quad (26)$$

In turn, arguments of the Legendre functions (hypergeometric functions) in this representation are expressions containing another Abelian function via the quantity $\sqrt{1 - \mathbf{J}}$, i. e., $\sum \gamma_k^2(x)$, by virtue of formula (24). This function is represented by its own rational fraction of Θ -functions of the type $\Theta(x\mathbf{U} + \mathbf{D})$. The nature of transcendental relations is therefore rather involved and still needs to be understood. The transcendency is preserved even when the Jacobian of curve (7) splits completely into elliptic curves, and the multidimensional Θ -functions become the elliptic θ -functions. Polynomials of type (16) then remain nondegenerate, and it is natural to expect the appearance of transcendental θ -identities.

The type of transcendency obtained above is related to the elliptic modular inversion because this inversion is generated by a solution of an algebraic equation with application of the elliptic modular functions. The general theta functions and constants appear in more-general methods (see, e. g., Umemura’s contribution to [12]), and it is therefore natural to expect the appearance of other types of transcendental identities.

In conclusion, we note that the simultaneous and transcendental unification of theta functions and hypergeometric series is in fact not unexpected, because a natural feature of ${}_2F_1$ series are their monodromy groups, which are in turn automorphisms of analytic automorphic functions. All such functions that are currently known are ratios of theta constants. Informally speaking, the “theta” geometry and “hypergeometry” are mutually inverse (they realize inversions of fractions of type (20)), and our transcendental identities reflect both this fact and the properties of functions of the type $\Theta(x\mathbf{U} + \mathbf{D})$. Namely, Abelian manifolds are parameterized as algebraic manifolds by polynomials in theta functions while linear sections of Jacobians appear from integrable equations. A

partial illustration is already provided by Jacobi equation (22) rewritten in the form

$$\tau \equiv i \frac{K' \left(\frac{\vartheta_3^2(\tau)}{\vartheta_2^2(\tau)} \right)}{K \left(\frac{\vartheta_3^2(\tau)}{\vartheta_2^2(\tau)} \right)} \pmod{\Gamma(2)} \quad \forall \tau \in \mathbb{H}^+.$$

Considering this equation in a sufficiently small vicinity of a point τ_0 , we can drop out a “not completely analytic” operation of bringing a point into the fundamental domain of the group, and this equation then becomes an exact analytic transcendental identity. We can proceed in the same way with the theta versions of our relations (25) and (26) interpreting them as functions of x .

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